Observations on the Pell Equation $y^2 = 11x^2 + 5$

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Abstract – The binary quadratic equation $y^2 = 11x^2 + 5$ is considered and a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a special pythagorean triangle is obtained.

Index Terms – Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.

1. INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is a non-square positive integer, has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1,2,3,4]. In [5], infinitely many pythogorean triangles in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation $y^2 = 3x^2 + 1$. In [6], a special pythogorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$. In [7], different patterns of infinitely many pythagorean triangles are obtained by employing the non-trivial solutions of $y^2 = 12x^2 + 1$. In this context, one may also refer [8-18]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2 = 11x^2 + 5$ representing a hyperbola. A few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a special pythagorean triangle is obtained.

2. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is

$$y^2 = 11x^2 + 5 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 1, y_0 = 4$$

To obtain the other solutions of (1), consider the pell equation

$$y^2 = 11x^2 + 1 \qquad (2)$$

whose initial solution is

$$\tilde{x}_0 = 3, \tilde{y}_0 = 10$$

The general solution of (2) is given by

$$\tilde{x}_n = \frac{g_n}{2\sqrt{11}}, \tilde{y}_n = \frac{f_n}{2}$$

where

$$f_n = \left(10 + 3\sqrt{11}\right)^{n+1} + \left(10 - 3\sqrt{11}\right)^{n+1},$$

$$g_n = \left(10 + 3\sqrt{11}\right)^{n+1} - \left(10 - 3\sqrt{11}\right)^{n+1},$$

n = -1, 0, 1, 2……

Applying Brahmagupta lemma between the solutions of (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{4}{2\sqrt{11}} g_n \qquad (3)$$

$$y_{n+1} = 2 f_n + \frac{11}{2\sqrt{11}} g_n \qquad (4)$$

Thus (3) and (4) represent the non-zero distinct integer solutions of (1).

The recurrence relations satisfied by the values of x_{n+1} and y_{n+1} are respectively.

$$x_{n+3} - 20x_{n+2} + x_{n+1} = 0, n = -1,0,1\ldots\ldots$$

$$y_{n+3} - 20y_{n+2} + y_{n+1} = 0, n = -1,0,1\ldots\ldots$$

A few numerical examples are given in the following Table: 2.1 below:

Table: 2.1 NUMERICAL EXAMPLES

n	x_{n+1}	y_{n+1}
-1	1	4
0	22	73
1	439	1456
2	8758	29047

2.1 A few interesting relations among the solutions are given below

1. $x_{n+1}-20x_{n+2}+x_{n+3}=0$
2. $110x_{n+1}-11x_{n+2}+33y_{n+1}=0$
3. $11x_{n+1}-110x_{n+2}+33y_{n+2}=0$
4. $110x_{n+1}-2189x_{n+2}+33y_{n+3}=0$
5. $199x_{n+1}-x_{n+3}+60y_{n+1}=0$
6. $10x_{n+1}-x_{n+2}+3y_{n+1}=0$
7. $33x_{n+1}+10y_{n+1}-y_{n+2}=0$
8. $660x_{n+1}+199y_{n+1}-y_{n+3}=0$
9. $x_{n+1}-x_{n+3}+6y_{n+2}=0$
10. $66x_{n+1}+10y_{n+1}-y_{n+2}=0$
11. $363x_{n+1}+2189y_{n+2}-110y_{n+3}=0$
12. $x_{n+1}-199x_{n+3}+60y_{n+3}=0$
13. $2189x_{n+2}-110x_{n+3}+33y_{n+1}=0$
14. $10x_{n+2}-x_{n+3}+3y_{n+2}=0$
15. $x_{n+2}-10x_{n+3}+3y_{n+3}=0$
16. $33x_{n+2}+y_{n+1}-10y_{n+2}=0$
17. $x_{n+1}-10x_{n+2}+3y_{n+2}=0$
18. $33x_{n+2}+10y_{n+2}-y_{n+3}=0$
19. $363x_{n+3}+110y_{n+1}-2189y_{n+2}=0$
20. $33x_{n+3}+y_{n+2}-10y_{n+3}=0$
21. $660x_{n+3}+y_{n+1}-199y_{n+3}=0$
22. $66x_{n+2}+y_{n+1}-y_{n+3}=0$
23. $y_{n+1}-20y_{n+2}+y_{n+3}=0$

2.2 Each of the following expression represents a cubic integer

i. $\frac{1}{15}\left[\left(8x_{3n+4}-146x_{3n+3}\right)+3\left(8x_{n+2}-146x_{n+1}\right)\right]$

ii. $\frac{1}{300}\left[\left(8x_{3n+5}-2912x_{3n+3}\right)+3\left(8x_{n+3}-2912x_{n+1}\right)\right]$

iii. $\frac{1}{5}\left[\left(8y_{3n+3}-22x_{3n+3}\right)+3\left(8y_{n+1}-22x_{n+1}\right)\right]$

iv. $\frac{1}{50}\left[\left(8y_{3n+4}-484x_{3n+3}\right)+3\left(8y_{n+2}-484x_{n+1}\right)\right]$

v. $\frac{1}{995}\left[\left(8y_{3n+5}-9658x_{3n+3}\right)+3\left(8y_{n+3}-9658x_{n+1}\right)\right]$

vi. $\frac{1}{15}\left[\left(146x_{3n+5}-2912x_{3n+4}\right)+3\left(146x_{n+3}-2912x_{n+2}\right)\right]$

vii. $\frac{1}{50}\left[\left(146y_{3n+3}-22x_{3n+4}\right)+3\left(146y_{n+1}-22x_{n+2}\right)\right]$

viii. $\frac{1}{5}\left[\left(146y_{3n+4}-484x_{3n+4}\right)+3\left(146y_{n+2}-484x_{n+2}\right)\right]$

ix. $\frac{1}{50}\left[\left(146y_{3n+5}-9658x_{3n+4}\right)+3\left(146y_{n+3}-9658x_{n+2}\right)\right]$

x. $\frac{1}{995}\left[\left(2912y_{3n+3}-22x_{3n+5}\right)+3\left(2912y_{n+1}-22x_{n+3}\right)\right]$

xi. $\frac{1}{50}\left[\left(2912y_{3n+4}-484x_{3n+5}\right)+3\left(2912y_{n+2}-484x_{n+3}\right)\right]$

xii. $\frac{1}{5}\left[\left(2912y_{3n+5}-9658x_{3n+5}\right)+3\left(2912y_{n+3}-9658x_{n+3}\right)\right]$

xiii. $\frac{1}{165}\left[\left(484y_{3n+3}-22y_{3n+4}\right)+3\left(484y_{n+1}-22y_{n+2}\right)\right]$

xiv. $\frac{1}{3300}\left[\left(9658y_{3n+3}-22y_{3n+5}\right)+3\left(9658y_{n+1}-22y_{n+3}\right)\right]$

xv. $\frac{1}{165}\left[\left(9658y_{3n+4}-484y_{3n+5}\right)+3\left(9658y_{n+2}-484y_{n+3}\right)\right]$

2.3 Each of the following expression represents a bi-quadratic integer

i. $\frac{1}{15^2}\left[\begin{array}{l}\left(120x_{4n+5}-2190x_{4n+4}\right)+\\ 4\left(8x_{n+2}-146x_{n+1}\right)^2-450\end{array}\right]$

ii. $\frac{1}{300^2}\begin{bmatrix}(2400x_{4n+6}-873600x_{4n+4})\\+4(8x_{n+3}-2912x_{n+1})^2-180000\end{bmatrix}$

iii. $\frac{1}{5^2}\left[(40y_{4n+4}-110x_{4n+4})+4(8y_{n+1}-22x_{n+1})^2-50\right]$

iv. $\frac{1}{50^2}\begin{bmatrix}(400y_{4n+5}-24200x_{4n+4})\\+4(8y_{n+2}-484x_{n+1})^2-5000\end{bmatrix}$

v. $\frac{1}{995^2}\begin{bmatrix}(7960y_{4n+6}-9609710x_{4n+4})\\+4(8y_{n+3}-9658x_{n+1})^2-1980050\end{bmatrix}$

vi. $\frac{1}{15^2}\begin{bmatrix}(2190x_{4n+6}-43680x_{4n+5})\\+4(146x_{n+3}-2912x_{n+2})^2-450\end{bmatrix}$

vii. $\frac{1}{50^2}\begin{bmatrix}(7300y_{4n+4}-1100x_{4n+5})\\+4(146y_{n+1}-22x_{n+2})^2-5000\end{bmatrix}$

viii. $\frac{1}{5^2}\begin{bmatrix}(730y_{4n+5}-2420x_{4n+5})\\+4(146y_{n+2}-484x_{n+2})^2-50\end{bmatrix}$

ix. $\frac{1}{50^2}\begin{bmatrix}(7300y_{4n+6}-482900x_{4n+5})\\+4(146y_{n+3}-9658x_{n+2})^2-50000\end{bmatrix}$

x. $\frac{1}{995^2}\begin{bmatrix}(2897440y_{4n+4}-21890x_{4n+6})\\+4(2912y_{n+1}-22x_{n+3})^2-1980050\end{bmatrix}$

xi. $\frac{1}{50^2}\begin{bmatrix}(145600y_{4n+5}-24200x_{4n+6})\\+4(2912y_{n+2}-484x_{n+3})^2-5000\end{bmatrix}$

xii. $\frac{1}{5^2}\begin{bmatrix}(14560y_{4n+6}-48290x_{4n+6})\\+4(2912y_{n+3}-9658x_{n+3})^2-50\end{bmatrix}$

xiii. $\frac{1}{165^2}\begin{bmatrix}(79860y_{4n+4}-3630y_{4n+5})\\+4(484y_{4n+4}-22y_{4n+5})^2-54450\end{bmatrix}$

xiv. $\frac{1}{3300^2}\begin{bmatrix}(31871400y_{4n+4}-72600y_{4n+6})\\+4(9658y_{n+1}-22y_{n+3})^2-21780000\end{bmatrix}$

xv. $\frac{1}{165^2}\begin{bmatrix}(1593570y_{4n+5}-79860y_{4n+6})\\+4(9658y_{n+2}-484y_{n+3})^2-54450\end{bmatrix}$

2.4 Each of the following expression represents a nasty number

- $\frac{6}{15}\left[30+8x_{2n+3}-146x_{2n+2}\right]$
- $\frac{6}{300}\left[600+8x_{2n+4}-2912x_{2n+2}\right]$
- $\frac{6}{5}\left[10+8y_{2n+2}-22x_{2n+2}\right]$
- $\frac{6}{50}\left[100+8y_{2n+3}-484x_{2n+2}\right]$
- $\frac{6}{995}\left[1990+8y_{2n+4}-9658x_{2n+2}\right]$
- $\frac{6}{15}\left[30+146x_{2n+4}-2912x_{2n+3}\right]$
- $\frac{6}{50}\left[100+146y_{2n+2}-22x_{2n+3}\right]$
- $\frac{6}{5}\left[10+146y_{2n+3}-484x_{2n+3}\right]$
- $\frac{6}{50}\left[100+146y_{2n+4}-9658x_{2n+3}\right]$
- $\frac{6}{995}\left[1990+2912y_{2n+2}-22x_{2n+4}\right]$
- $\frac{6}{50}\left[100+2912y_{2n+3}-484x_{2n+4}\right]$
- $\frac{6}{5}\left[10+2912y_{2n+4}-9658x_{2n+4}\right]$
- $\frac{6}{165}\left[330+484y_{2n+2}-22y_{2n+2}\right]$
- $\frac{6}{3300}\left[6600+9658y_{2n+2}-22y_{2n+4}\right]$
- $\frac{6}{165}\left[330+9658y_{2n+3}-484y_{2n+4}\right]$

2.5 Each of the following expression represents a quintic integer

i. $\frac{1}{(15)^3}\begin{bmatrix}1800x_{5n+6}-32850x_{5n+5}\\+5(8x_{n+2}-146x_{n+2})^3\\-9000x_{n+2}+164250x_{n+1}\end{bmatrix}$

ii. $$\frac{1}{(300)^3}\left[\begin{array}{l}720000x_{5n+7}-262080000x_{5n+5}\\+5(8x_{n+3}-2912x_{n+1})^3-\\3600000x_{n+3}+1310400000x_{n+1}\end{array}\right]$$

iii. $$\frac{1}{(5)^3}\left[\begin{array}{l}200y_{5n+5}-500x_{5n+5}+\\5(8y_{n+1}-22x_{n+1})^3\\-1000y_{n+1}+2500x_{n+1}\end{array}\right]$$

iv. $$\frac{1}{(50)^3}\left[\begin{array}{l}20000y_{5n+6}-1210000x_{5n+5}\\+5(8y_{n+2}-484x_{n+1})^3\\-100000y_{n+2}+6050000x_{n+1}\end{array}\right]$$

v. $$\frac{1}{(995)^3}\left[\begin{array}{l}7920200y_{5n+7}-9561661450x_{5n+5}\\+5(8y_{n+3}-9658x_{n+1})^3\\-396010000y_{n+3}-47808307250x_{n+1}\end{array}\right]$$

vi. $$\frac{1}{(15)^3}\left[\begin{array}{l}32850x_{5n+7}-655200x_{5n+6}\\+5(146x_{n+3}-2912x_{n+2})^3\\-164250x_{n+3}+3276000x_{n+2}\end{array}\right]$$

vii. $$\frac{1}{(50)^3}\left[\begin{array}{l}365000y_{5n+5}-55000x_{5n+6}\\+5(146y_{n+1}-22x_{n+2})^3\\-1825000y_{n+1}+275000x_{n+2}\end{array}\right]$$

viii. $$\frac{1}{(5)^3}\left[\begin{array}{l}3650y_{5n+6}-12100x_{5n+6}\\+5(416y_{n+2}-484x_{n+2})^3\\-18250y_{n+2}+60500x_{n+2}\end{array}\right]$$

ix. $$\frac{1}{(50)^3}\left[\begin{array}{l}365000y_{5n+7}-24145000x_{5n+6}\\+5(146y_{n+3}-9658x_{n+2})^3\\-1825000y_{n+3}+120725000x_{n+2}\end{array}\right]$$

x. $$\frac{1}{(995)^3}\left[\begin{array}{l}2882952800y_{5n+5}-21780550x_{5n+7}\\+5(2912y_{n+1}-22x_{n+3})^3\\-14414764000y_{n+1}+108902750x_{n+3}\end{array}\right]$$

xi. $$\frac{1}{(50)^3}\left[\begin{array}{l}7280000y_{5n+6}-1210000x_{5n+7}\\+5(2912y_{n+2}-484x_{n+3})^3\\-36400000y_{n+2}+6050000x_{n+3}\end{array}\right]$$

xii. $$\frac{1}{(5)^3}\left[\begin{array}{l}72800y_{5n+7}-241450x_{5n+7}\\+5(2912y_{n+3}-9658x_{n+3})^3\\-364000y_{n+3}+1207250x_{n+3}\end{array}\right]$$

xiii. $$\frac{1}{(165)^3}\left[\begin{array}{l}13176900y_{5n+5}-598950y_{5n+6}\\+5(484y_{n+1}-22y_{n+2})^3\\-65884500y_{n+1}+2994750y_{n+2}\end{array}\right]$$

xiv. $$\frac{1}{(3300)^3}\left[\begin{array}{l}105175620000y_{5n+5}-239580000y_{5n+7}\\+5(9658y_{n+1}-22y_{n+3})^3\\-525878100000y_{n+1}+1197900000y_{n+1}\end{array}\right]$$

xv. $$\frac{1}{(165)^3}\left[\begin{array}{l}262939050y_{5n+6}-13176900y_{5n+7}\\+5(9658y_{n+2}-484y_{n+3})^3\\-1314695250y_{n+2}+65884500y_{n+3}\end{array}\right]$$

2.6 Remarkable Observations:

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in Table: 2.6.1 below.

Table : 2.6.1 HYPERBOLAS

S.NO	HYPERBOLAS	(X_n, Y_n)
1.	$11X_n^2 - Y_n^2 = 9900$	$X_n = 8x_{n+2} - 146x_{n+1}$ $Y_n = 484x_{n+1} - 22x_{n+2}$
2.	$11X_n^2 - Y_n^2 = 3960000$	$X_n = 8x_{n+3} - 2912x_{n+1}$ $Y_n = 9658x_{n+1} - 22x_{n+3}$
3.	$11X_n^2 - Y_n^2 = 1100$	$X_n = 8y_{n+1} - 22x_{n+1}$ $Y_n = 88x_{n+1} - 22y_{n+1}$
4.	$11X_n^2 - Y_n^2 = 110000$	$X_n = 8y_{n+2} - 484x_{n+1}$ $Y_n = 1606x_{n+1} - 22y_{n+2}$
5.	$11X_n^2 - Y_n^2 = 43561100$	$X_n = 8y_{n+3} - 9658x_{n+1}$ $Y_n = 32032x_{n+1} - 22y_{n+3}$
6.	$11X_n^2 - Y_n^2 = 9900$	$X_n = 146x_{n+3} - 2912x_{n+2}$ $Y_n = 9658x_{n+2} - 484x_{n+3}$
7.	$11X_n^2 - Y_n^2 = 110000$	$X_n = 146y_{n+1} - 22x_{n+2}$ $Y_n = 88x_{n+2} - 484y_{n+1}$

8.	$11X_n^2 - Y_n^2 = 1100$	$X_n = 146y_{n+2} - 484x_{n+2}$ $Y_n = 1606x_{n+2} - 484y_{n+2}$
9.	$11X_n^2 - Y_n^2 = 110000$	$X_n = 146y_{n+3} - 9658x_{n+2}$ $Y_n = 32032x_{n+2} - 484y_{n+3}$
10.	$11X_n^2 - Y_n^2 = 43561100$	$X_n = 2912y_{n+1} - 22x_{n+3}$ $Y_n = 88x_{n+3} - 9658y_{n+1}$
11.	$11X_n^2 - Y_n^2 = 110000$	$X_n = 2912y_{n+2} - 484x_{n+3}$ $Y_n = 1606x_{n+3} - 9658y_{n+2}$
12.	$11X_n^2 - Y_n^2 = 1100$	$X_n = 2912y_{n+3} - 9658x_{n+3}$ $Y_n = 32032x_{n+3} - 9658y_{n+3}$
13.	$11X_n^2 - Y_n^2 = 1197900$	$X_n = 484y_{n+1} - 22y_{n+2}$ $Y_n = 88y_{n+2} - 1606y_{n+1}$
14.	$11X_n^2 - Y_n^2 = 479160000$	$X_n = 9658y_{n+1} - 22y_{n+3}$ $Y_n = 88y_{n+3} - 32032y_{n+1}$
15.	$11X_n^2 - Y_n^2 = 1197900$	$X_n = 9658y_{n+2} - 484y_{n+3}$ $Y_n = 1606y_{n+3} - 32032y_{n+2}$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table: 2.6.2 below:

Table: 2.6.2 PARABOLAS

S.NO	PARABOLAS	(X_n, Y_n)
1.	$165X_n - Y_n^2 = 9900$	$X_n = 8x_{2n+3} - 146x_{2n+2} + 30$ $Y_n = 484x_{n+1} - 22x_{n+2}$
2.	$3300X_n - Y_n^2 = 3960000$	$X_n = 8x_{2n+4} - 2912x_{2n+2} + 600$ $Y_n = 9658x_{n+1} - 22x_{n+3}$
3.	$55X_n - Y_n^2 = 1100$	$X_n = 8y_{2n+2} - 22x_{2n+2} + 10$ $Y_n = 88x_{n+1} - 22y_{n+1}$
4.	$550X_n - Y_n^2 = 110000$	$X_n = 8y_{2n+3} - 484x_{2n+2} + 100$ $Y_n = 1606x_{n+1} - 22y_{n+2}$
5.	$10945X_n - Y_n^2 = 43561100$	$X_n = 8y_{2n+4} - 9658x_{2n+2} + 1990$ $Y_n = 32032x_{n+1} - 22y_{n+2}$
6.	$165X_n - Y_n^2 = 9900$	$X_n = 146x_{2n+4} - 2912x_{2n+3} + 30$ $Y_n = 9658x_{n+2} - 484x_{n+3}$
7.	$550X_n - Y_n^2 = 110000$	$X_n = 146y_{2n+2} - 22x_{2n+3} + 100$ $Y_n = 88x_{n+2} - 484y_{n+1}$
8.	$55X_n - Y_n^2 = 4400$	$X_n = 146y_{2n+3} - 484x_{2n+3} + 10$ $Y_n = 1606x_{n+2} - 484y_{n+2}$
9.	$550X_n - Y_n^2 = 110000$	$X_n = 146y_{2n+4} - 9658x_{2n+3} + 100$ $Y_n = 32032x_{n+2} - 484y_{n+3}$
10.	$10945X_n - Y_n^2 = 43561100$	$X_n = 2912y_{2n+2} - 22x_{2n+2} + 1990$ $Y_n = 88x_{n+3} - 9658y_{n+1}$
11.	$550X_n - Y_n^2 = 110000$	$X_n = 2912y_{2n+3} - 484x_{2n+4} + 100$ $Y_n = 1606x_{n+3} - 9658y_{n+2}$
12.	$55X_n - Y_n^2 = 1100$	$X_n = 2912y_{2n+4} - 9658x_{2n+4} + 10$ $Y_n = 32032x_{n+3} - 9658y_{n+3}$
13.	$1815X_n - Y_n^2 = 1197900$	$X_n = 484y_{2n+2} - 22y_{2n+3} + 330$ $Y_n = 88y_{n+2} - 1606y_{n+1}$
14.	$36300X_n - Y_n^2 = 479160000$	$X_n = 9658y_{2n+2} - 22y_{2n+4} + 6600$ $Y_n = 88y_{n+3} - 32032y_{n+1}$
15.	$1815X_n - Y_n^2 = 1197900$	$X_n = 9658y_{2n+3} - 484y_{2n+4} + 330$ $Y_n = 1606y_{n+3} - 32032y_{n+2}$

2.7 Generation of Pythagorean triangle

2.7.1 Let p, q be the non-zero distinct integers such that

$$p = x_{n+1} + y_{n+1}, \; q = y_{n+1}$$

Note that $p > q > 0$ treat p, q as the generators of Pythagorean triangle $T(X,Y,Z)$ where

$$X = 2pq \ , \ Y = p^2 - q^2, \ Z = p^2 + q^2, \ p > q > 0$$

Let A, P represents the area and perimeter of Pythagorean triangle. Then the following results are observed.

(i) $2X - 11Y + 9Z = -10$

(ii) $\frac{2A}{P} = x_{n+1} * y_{n+1}$

(iii) $X + Y - \frac{4A}{P}$ is written as the sum of two squares.

(iv) $3\left(X - \frac{4A}{P} \right)$ is a Nasty number.

(v) $3(Z - Y)$ is a Nasty number.

2.7.2 Let p, q be the non-zero distinct integers such that

$$p = x_{n+1} + y_{n+1}, \ q = y_{n+1}$$

Note that $p > q > 0$ treat p, q as the generators of Pythagorean triangle $T(X,Y,Z)$ where

$$X = 2pq, \ Y = p^2 - q^2, \ Z = p^2 + q^2, \ p > q > 0$$

Let A, P represents the area and perimeter of Pythagorean triangle. In this case, the corresponding Pythagorean triangle satisfies the relation $Y - 22X + 21Z = -10$

3. CONCLUSION

To conclude, one may search for other choices of positive Pell equations for finding their integer solutions with suitable properties.

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